University of Greifswald (July, 1997)

hep-th/9707040

to appear in:

J. Mathematical Physics (1998)

arXiv:hep-th/9707040v2 10 Apr 1998

Green functions of higher-order differential operators

Ivan G. Avramidi [1] [2]

Department of Mathematics, University of Greifswald
F.-L.-Jahnstr. 15a, D–17489 Greifswald, Germany

and

Department of Mathematics, The University of Iowa
14 MacLean Hall, Iowa City, IA 52242-1419, USA

February 1, 2008

The Green functions of the partial differential operators of even order acting on smooth sections of a vector bundle over a Riemannian manifold are investigated via the heat kernel methods. We study the resolvent of a special class of higher-order operators formed by the products of second-order operators of Laplace type defined with the help of a unique Riemannian metric but with different bundle connections and potential terms. The asymptotic expansion of the Green functions near the diagonal is studied in detail in any dimension. As a by-product a simple criterion for the validity of the Huygens principle is obtained. It is shown that all the singularities as well as the non-analytic regular parts of the Green functions of such high-order operators are expressed in terms of the usual heat kernel coefficients a_k for a special Laplace type second-order operator.

PACS: 02.30.Jr; 02.40.Vh; 02.90.+p; 04.62.+v

[1] On leave of absence from Research Institute for Physics, Rostov State University, Stachki 194, 344104 Rostov-on-Don, Russia

[2] Electronic-mail: iavramid@math.uiowa.edu

I INTRODUCTION

The Green functions of partial differential operators are of great importance in mathematical physics and quantum theory [1, 2, 3, 4]. In particular, the singularities of the Green functions on the diagonal play a crucial role in the renormalization procedure of quantum field theory [5].

In this paper we study a special class of differential operators of higher orders that appear, in particular, in higher-derivative field theories (for example, in higher-derivative quantum gravity [6]). Namely, we consider differential operators of higher *even* orders, $2N$, of the form

$$H = (F_N + m^2) \cdots (F_2 + m^2)(F_1 + m^2), \tag{1}$$

where F_i, $(i = 1, \dots, N)$ are Laplace type operators, i.e. second-order operators with the scalar leading symbol determined by the metric of the manifold

$$\sigma_L(F_i) = |\xi|^2, \tag{2}$$

where $|\xi|^2 \equiv g^{\mu\nu}(x)\xi_\mu\xi_\nu$, and m is some constant. In the next section we give precise definition of Laplace type operators and describe their properties. Obviously, the operator H also has a scalar leading symbol

$$\sigma_L(H) = |\xi|^{2N}, \tag{3}$$

and is a particular case of the so called minimal higher-derivative operators.

We would like to stress from the beginning that the operators F_i, even if of the same Laplace type, are *different* and, in general, *do not commute*. They only have the same leading, second-derivatives, part, but differ in the first and zero order parts.

The Green function, or the resolvent kernel, $G_{H-\lambda}(x, x')$ of the operator H is the kernel of the Green, or resolvent, operator $G_{H-\lambda} = (H - \lambda)^{-1}$ and is defined by requring it to satisfy the differential equation

$$(H - \lambda)G_{H-\lambda}(x, x') = \delta(x, x'), \tag{4}$$

with H acting on the first argument of the Green function and $\delta(x, x')$ being the covariant delta-distribution.

In the present paper we will consider mainly the case of compact complete manifolds without boundary and assume the metric of the manifold to be positive definite. Then the Laplace type operators F_i as well as the operator H are *elliptic* and eq. (4) defines a unique Green function. We will also assume for simplicity the constant m to be sufficiently large so that all operators $(F_i + m^2)$ and also H are *positive*.

However, we would like to make some remarks concerning this subject:

i) First, for manifolds with boundary (in elliptic case) one has to impose additionally some suitable boundary conditions to make the solution of the eq. (4) unique. However, the singularities of the Green function near the diagonal and the asymptotic expansions of it as $m \to \infty$ do not depend on the boundary conditions.

ii) Second, when the metric of the manifold has Minkowski type signature, $\operatorname{sign} g = (-, +, \dots, +)$, the Laplace type operators F_i are *hyperbolic*. In this case the Green function can be fixed by the Wick rotation, i.e. the analytic continuation from the Euclidean case, which is equivalent to adding an infinitesimal negative imaginary part to the constant m^2, $m^2 \to m^2 - i\varepsilon$. In other words, we consider the *Feynman propagators*. The formulas for the hyperbolic case can be obtained just by analytic continuation.

II LAPLACE TYPE OPERATORS

Let (M, g) be a smooth Riemannian manifold of dimension d with a metric g. To simplify the exposition we assume the manifold M to be complete and compact, i.e. without boundary, $\partial M = \emptyset$, and the metric g to be positive definite.

Let $V(M)$ be a smooth vector bundle over the manifold M, $\mathrm{End}\,(V)$ be the bundle of all smooth endomorphisms of the vector bundle V, and $C^\infty(M, V)$ and $C^\infty(M, \mathrm{End}\,(V))$ be the spaces of all smooth sections of the vector bundles V and $\mathrm{End}\,(V)$. We assume, as usual, the vector bundle V to be Hermitian, i.e. there is a Hermitian pointwise fibre scalar product. Then the dual vector bundle V^* is naturally identified with V and a natural L^2 inner product is defined using the invariant Riemannian volume element $d\mathrm{vol}\,(x)$ on the manifold M. The completion of $C^\infty(M, V)$ in this norm defines the Hilbert space of square integrable sections of the vector bundle $L^2(M, V)$.

Let, further, ∇ be a connection, or covariant derivative, on the vector bundle V which is compatible with the Hermitian metric on the vector bundle V. Denoting by T^*M the cotangent bundle we define the tensor product connection on the tensor product bundle $T^*M \otimes V$ by means of the Levi-Civita connection. Similarly, we extend the connection ∇ with the help of the Levi-Civita connection to $C^\infty(M, V)$-valued tensors of all orders and denote it just by ∇.

Usually there is no ambiguity and the precise meaning of the covariant derivative is always clear from the nature of the object it is acting on.

Let, further, tr_g denote the contraction of sections of the bundle $T^*M \otimes T^*M \otimes V$ with the metric on the cotangent bundle, and $Q \in C^\infty(M, \mathrm{End}(V))$ be a a smooth Hermitian section of the endomorphism bundle. Then we define the generalized Laplacian

$$\Box = \mathrm{tr}_g \nabla\nabla \tag{5}$$

and a *Laplace type* differential operator $F:\ C^\infty(M, V) \to C^\infty(M, V)$ by

$$F = -\Box + Q. \tag{6}$$

Let x^μ, $(\mu = 1, 2, \dots, d)$, be a system of local coordinates and ∂_μ and dx^μ be the local coordinate frames for the tangent and the cotangent bundles. We adopt the notation that the Greek indices label the tensor components with respect to local coordinate frame and range from 1 through d. Besides, a summation is always carried out over repeated indices. Let $g_{\mu\nu} = (\partial_\mu, \partial_\nu)$ be the metric on the tangent bundle, $g^{\mu\nu} = (dx^\mu, dx^\nu)$ be the metric

on the cotangent bundle, $g = \det g_{\mu\nu}$, and $\mathcal{A}_\mu$ be the connection $1-$form on the vector bundle V. Then the generalized Laplacian reads

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{-1/2}(\partial_\mu + \mathcal{A}_\mu)g^{1/2}g^{\mu\nu}(\partial_\nu + \mathcal{A}_\nu). \tag{7}$$

It is worth noting that every second-order differential operator with a scalar

leading symbol given by the metric tensor is of Laplace type and can be put in the form (6) by choosing the appropriate connection 1-form $\mathcal{A}$ and the endomorphism Q.

The operator F has a positive leading symbol and is elliptic. It is symmetric with respect to the natural L^2 inner product. Moreover, the operator F is essentially self-adjoint, i.e. there is a unique

self-adjoint extension $\bar{F}$ of the operator F. However, we will not be very careful about distinguishing between the operator F and its closure $\bar{F}$, and will simply say that the operator F is elliptic and self-adjoint.

II.1 Heat kernel of a Laplace type operator

Hence, the operator $U_F(t) = \exp(-tF)$ for $t > 0$ is well defined as a *bounded* operator on the Hilbert space $L^2(M, V)$ of square integrable sections of the vector bundle V. This operator forms a one-parameter semi-group. The kernel $U_F(t|x, x')$ of this operator satisfies the heat equation and is called the heat kernel. For $t > 0$ the heat kernel is a smooth section of the external tensor product of the vector bundles V and V^* over the tensor product manifold $M \times M$.

In the following we are going to study the Green function and the heat kernel only locally (like in [7]), i.e. in the neighbourhood of the diagonal of $M \times M$. We will keep a point x' of the manifold fixed and consider a small geodesic ball, i.e. a small neighbourhood of the point x': $B_{x'} = \{x \in M | r(x, x') < \varepsilon\}$, $r(x, x')$ being the geodesic distance between the points x and x'. We will take the radius of the ball sufficiently small, so that each point x of the ball of this neighbourhood can be connected by a unique geodesic with the point x'. This can be always done if the size of the ball is smaller than the injectivity radius of the manifold at x', $\varepsilon < r_{\rm inj}(x')$.

Let $\sigma(x, x')$ be the geodetic interval, also called world function, defined as one half the square of the length of the geodesic connecting the points x and x'

$$\sigma(x, x') = \frac{1}{2}r^2(x, x'). \tag{8}$$

The first derivatives of this function with respect to x and x' define the tangent vector fields to the geodesic at the end points x and x' and the determinant of the mixed second derivatives defines a so called Van Vleck-Morette determinant [2, 8, 9]

$$\Delta(x, x') = g^{-1/2}(x)\det\left(-\nabla_\mu\nabla'_{\nu'}\sigma(x, x')\right)g^{-1/2}(x'). \tag{9}$$

Let, finally, $\mathcal{P}(x, x')$ denote the paralell transport operator along the geodesic from the point x' to the point x. It is a section of the external tensor product of the vector bundle V and V^* over $M \times M$, or, in other words, it is an endomorphism from the fiber of V over

x' to the fiber of V over x. Near the diagonal of $M\times M$ all these two-point functions are smooth single-valued functions of the coordinates of the points x and x'. To simplify the consideration, one can assume that these functions are *analytic* in the ball $B_{x'}$.

The heat kernel of a Laplace type operator is well described by factorizing out the semi-classical factor [2, 6, 7]

$$U_F(t|x,x') = (4\pi t)^{-d/2}\Delta^{1/2}(x,x')\exp\left(-\frac{1}{2t}\sigma(x,x')\right)\Omega_F(t|x,x'). \tag{10}$$

The function $\Omega_F(t|x,x')$ is called the *transfer function* of the operator F. Obviously,

$$U_{F+m^2}(t) = e^{-tm^2}U_F(t), \qquad \Omega_{F+m^2}(t) = e^{-tm^2}\Omega_F(t). \tag{11}$$

It can be proved [7] that for a positive operator $F+m^2$ the transfer function $\Omega_{F+m^2}(t)$ can be presented in form of an inverse Mellin transform (we slightly change notation here in comparison to [7])

$$\Omega_{F+m^2}(t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} dq\, t^q\Gamma(-q)b_{F+m^2}(q), \tag{12}$$

where c is a negative constant. The function $b_{F+m^2}(q)$ is an *entire* function of q, so that the integrand has simple poles at the positive integer points $q=k$, $(k=0,1,\dots)$. The function $b_{F+m^2}(q|x,x')$ satisfies a functional-differential equation [7]

$$\left(1+\frac{1}{q}D\right)b_{F+m^2}(q) = L\,b_{F+m^2}(q-1), \tag{13}$$

where

$$L = \Delta^{-1/2}F\Delta^{1/2}+m^2 = -\Delta^{-1/2}\,\Box\,\Delta^{1/2}+Q+m^2, \tag{14}$$

$$D = \sigma^\mu\nabla_\mu, \qquad \sigma_\mu = \nabla_\mu\sigma, \tag{15}$$

and the initial condition

$$b_{F+m^2}(0|x,x') = \mathcal{P}(x,x'). \tag{16}$$

This equation (together with the condition of analyticity and some asymptotic condition at $q\to\text{const}\pm i\infty$) enables one to compute $b_{F+m^2}(q)$ if one fixes its initial value at some arbitrary point $q=q_0$. (For more details, see [7]). The initial condition at the origin produces the values of $b_{F+m^2}(k)$ at positive integer points, the initial condition at the point $q=1/2$ would give the values of $b_{F+m^2}(q)$ at all half-integer positive points $b_{F+m^2}(k+1/2)$, etc. Differentiating the eq. (13) with respect to q one obtains another recursion

$$\left(1+\frac{1}{q}D\right)b'_{F+m^2}(q) = L\,b'_{F+m^2}(q-1)+\frac{1}{q^2}Db_{F+m^2}(q), \tag{17}$$

where

$$b'_{F+m^2}(q) = \frac{\partial}{\partial q}b_{F+m^2}(q), \tag{18}$$

which enables one to compute the derivatives of the function $b_{F+m^2}(q)$ at positive integer points if one fixes its value $b'_{F+m^2}(0)$.

Moving the contour of integration to the right and taking into account that the residues of the gamma-function $\Gamma(-q)$ at the points $q=k$ equal $(-1)^k/k!$ one obtains [7]

$$\Omega_{F+m^2}(t)=\sum_{k=0}^{n-1}\frac{(-t)^k}{k!}a_k(F+m^2)+\frac{1}{2\pi i}\int\limits_{c_n-i\infty}^{c_n+i\infty}dq\,t^q\Gamma(-q)b_{F+m^2}(q), \tag{19}$$

where

$$a_k(F+m^2)\equiv b_{F+m^2}(k)=\left(-\frac{\partial}{\partial t}\right)^k\Omega_{F+m^2}(t)\Big|_{t=0}, \tag{20}$$

n is an arbitrary positive integer, $n\ge 1$, and c_n ranges in the interval $n-1<c_n<n$. The coefficients $a_k(F+m^2)$ are the famous Hadamard-Minakshisundaram-De Witt-Seeley (HMDS)-coefficients [1, 10, 2, 11] to the operator $F+m^2$. They can be computed in form of covariant Taylor series from the recursion relations that are obtained from the equation (13) (by putting $q\to k$) with the initial condition $a_0(F+m^2)=\mathcal{P}$. The diagonal values of the HMDS-coefficients are known in general case up to a_4 [12, 6, 13, 7, 14]. In some particular cases, e.g. in flat space, there are results for higher-order coefficients. For a review of the methods for calculation of HMDS-coefficients

and further references see [15, 16].

Taking into account eq. (11) we find from (20)

$$a_k(F+m^2)=\sum_{n=0}^{k}\binom{k}{n}m^{2(k-n)}a_n(F). \tag{21}$$

Moreover, one can obtain an asymptotic expansion of the function $b_{F+m^2}(q)$ as $m\to\infty$ [7]

$$b_{F+m^2}(q)\sim\sum_{n\ge 0}\frac{\Gamma(q+1)}{n!\Gamma(q-n+1)}m^{2(q-n)}a_n(F). \tag{22}$$

From (19) there follows an asymptotic expansion of the transfer function as $t\to 0$ (called Schwinger-De Witt expansion in the physical literature) [2, 3, 6, 7, 4]

$$\Omega_{F+m^2}(t)\sim\sum_{k\ge 0}\frac{(-t)^k}{k!}a_k(F+m^2)=e^{-tm^2}\sum_{k\ge 0}\frac{(-t)^k}{k!}a_k(F). \tag{23}$$

Using this equation one obtains from (10) the well known formula for the asymptotic expansion of the functional trace of the heat kernel

$$\mathrm{Tr}_{L^2}\exp(-tF)\sim(4\pi)^{-d/2}t^{-d/2}\sum_{k\ge 0}\frac{(-t)^k}{k!}A_k(F), \tag{24}$$

where

$$A_k(F)=\mathrm{Tr}_{L^2}a_k(F)=\int\limits_M d\mathrm{vol}\,(x)\,\mathrm{tr}_V a_k(F|x,x), \tag{25}$$

with tr_V being the fiber trace.

II.2 Green function of a Laplace type operator

The Laplace type operator F is a self-adjoint elliptic operator with a real spectrum bounded from below. Thus for sufficiently large m the operator $(F+m^2)$ is positive and its Green operator, or the resolvent operator, $G_{F+m^2} = (F+m^2)^{-1}$ is a bounded operator. It can be defined by

$$G_{F+m^2} = \int\limits_0^\infty dt\, e^{-tm^2} \exp(-tF). \tag{26}$$

The Green function of the operator $(F+m^2)$ is obtained by the kernel form of this equation

$$G_{F+m^2} = (4\pi)^{-d/2}\Delta^{1/2} \int\limits_0^\infty dt\, t^{-d/2} \exp\left(-tm^2 - \frac{\sigma}{2t}\right) \Omega_F(t). \tag{27}$$

This integral converges at the infinity for sufficiently large m. It also converges at $t=0$ outside the diagonal, i.e. for $\sigma \neq 0$. By using the ansatz (10) and (12) for the heat kernel one obtains a corresponding ansatz for the Green function

$$G_{F+m^2} = (4\pi)^{-d/2}\Delta^{1/2} \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} dq \left(\frac{\sigma}{2}\right)^{q+1-d/2} \Gamma(-q)\Gamma(-q-1+d/2) b_{F+m^2}(q), \tag{28}$$

where $c < -1/2$.

This ansatz is especially useful for studying the singularities of the Green function, or more general, for constructing the Green function as a power series in σ. The integrand in (28) is again a meromorphic function. However, contrary to (12), we have now a more complicated structure of the poles. There are always poles at the points $q = k$ and $q = k - 1 + d/2$, $(k = 0, 1, 2, \ldots)$. Here one has to distinguish between odd and even dimensions. In odd dimensions, the poles are at the points $q = k$ and $q = k + [d/2] - 1/2$ and are *simple*, whereas in even dimension there are simple poles at $q = 0, 1, 2, \ldots, d/2-2$ and *double* poles at the points $q = k + d/2 - 1$.

Moving the contour of integration in (28) to the right one can obtain an expansion of the Green function in powers of σ (Hadamard series). Generally, we obtain

$$G_{F+m^2} = G^{\rm sing}_{F+m^2} + G^{\rm non-anal}_{F+m^2} + G^{\rm reg}_{F+m^2}. \tag{29}$$

Here $G^{\rm sing}_{F+m^2}$ is the singular part which is polynomial in the inverse powers of $\sqrt{\sigma}$

$$G^{\rm sing}_{F+m^2} = (4\pi)^{-d/2}\Delta^{1/2} \sum_{k=0}^{[(d+1)/2]-2} \frac{(-1)^k}{k!} \Gamma(d/2-k-1) \left(\frac{2}{\sigma}\right)^{d/2-k-1} a_k(F+m^2), \tag{30}$$

For the rest we get in *odd* dimensions

$$G^{\rm non-anal}_{F+m^2} + G^{\rm reg}_{F+m^2}$$

$$
\begin{aligned}
&= (-1)^{\frac{d-1}{2}}(4\pi)^{-\frac{d}{2}}\Delta^{\frac{1}{2}}\sum_{k=0}^{n-(d+1)/2}\frac{\pi}{\Gamma\left(k+\frac{d+1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}\left(\frac{\sigma}{2}\right)^{k+1/2}a_{k+\frac{d-1}{2}}(F+m^2)\\
&+(-1)^{(d+1)/2}(4\pi)^{-d/2}\Delta^{1/2}\sum_{k=0}^{n-(d+1)/2}\frac{\pi}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^{k}b_{F+m^2}(k-1+d/2)\\
&+(4\pi)^{-d/2}\Delta^{1/2}\frac{1}{2\pi i}\int\limits_{c_n-i\infty}^{c_n+i\infty}dq\,\left(\frac{\sigma}{2}\right)^{q+1-d/2}\Gamma(-q)\Gamma(-q-1+d/2)b_{F+m^2}(q),
\end{aligned}
\tag{31}
$$

where $n-1<c_n<n-1/2$ and $n>(d-1)/2$. Thus, by putting $n\to\infty$ we recover herefrom the Hadamard power series in σ for *odd* $d=1,3,5,\dots$

$$
G^{\rm non-anal}_{F+m^2}\sim(-1)^{\frac{d-1}{2}}(4\pi)^{-\frac{d}{2}}\Delta^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\pi}{\Gamma\left(k+\frac{d+1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}\left(\frac{\sigma}{2}\right)^{k+1/2}a_{k+\frac{d-1}{2}}(F+m^2) \tag{32}
$$

$$
G^{\rm reg}_{F+m^2}\sim(-1)^{(d+1)/2}(4\pi)^{-d/2}\Delta^{1/2}\sum_{k=0}^{\infty}\frac{\pi}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^{k}b_{F+m^2}(k-1+d/2). \tag{33}
$$

In *even* dimensions, the point is more subtle due to the presence of double poles. Moving the contour in (28) to the right and calculating the contribution of the residues at the simple and double poles we obtain

$$
\begin{aligned}
&G^{\rm non-anal}_{F+m^2}+G^{\rm reg}_{F+m^2}\\
&=(-1)^{d/2-1}(4\pi)^{-d/2}\Delta^{1/2}\log\left(\frac{\mu^2\sigma}{2}\right)\sum_{k=0}^{n-1}\frac{1}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^{k}a_{k-1+d/2}(F+m^2)\\
&+(-1)^{d/2-1}(4\pi)^{-d/2}\Delta^{1/2}\sum_{k=0}^{n-1}\frac{1}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^{k}\\
&\times\Big\{b'_{F+m^2}(k-1+d/2)-\Big[\log\mu^2+\Psi(k+1)+\Psi(k+d/2)\Big]a_{k-1+\frac{d}{2}}(F+m^2)\Big\}\\
&+(4\pi)^{-d/2}\Delta^{1/2}\frac{1}{2\pi i}\int\limits_{c_n-i\infty}^{c_n+i\infty}dq\,\left(\frac{\sigma}{2}\right)^{q+1-d/2}\Gamma(-q)\Gamma(-q-1+d/2)b_{F+m^2}(q),
\end{aligned}
\tag{34}
$$

where μ is an arbitrary mass parameter introduced to preserve dimensions, $n-1<c_n<n$ and $\Psi(z)=(d/dz)\log\Gamma(z)$. If we let $n\to\infty$ we obtain the Hadamard expansion of the Green function for *even* $d=2,4,\dots$

$$
G^{\rm non-anal}_{F+m^2}\sim(-1)^{d/2-1}(4\pi)^{-d/2}\Delta^{1/2}\log\left(\frac{\mu^2\sigma}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^{k}a_{k-1+d/2}(F+m^2) \tag{35}
$$

$$
\begin{aligned}
G^{\rm reg}_{F+m^2} \sim\ & (-1)^{d/2-1}(4\pi)^{-d/2}\Delta^{1/2}\sum_{k=0}^\infty \frac{1}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^k \Big\{ b'_{F+m^2}(k-1+d/2) \\
& - \Big[\log\mu^2 + \Psi(k+1) + \Psi(k+d/2)\Big]\, a_{k-1+\frac{d}{2}}(F+m^2)\Big\}
\end{aligned}
\tag{36}
$$

Note that the singular part (which is a polynomial in inverse powers of $\sqrt{\sigma}$) and the *non-analytical* parts (proportional to $\sqrt{\sigma}$ and $\log\sigma$) are expressed *uniquely only in terms of the local HMDS-coefficients* $a_k(F+m^2)$, whereas the regular analytical part contains the values of the function $b_{F+m^2}(q)$ at half-integer positive points and the derivatives of the function $b_{F+m^2}(q)$ at integer positive points, which are *not* expressible in terms of the local information. These objects are global and cannot be expressed further in terms of the local HMDS-coefficients. However, they can be computed from the eqs. (13) and (17) almost by the same algorithm as the HMDS-coefficients in terms of the value of the function $b(q)$ at some fixed q_0 (see [7]).

The regular part of the Green function has a well defined diagonal value and the functional trace. It reads in odd dimensions ($d = 1, 3, 5, \ldots$):

$$
\begin{aligned}
{\rm Tr}_{L^2}\, G^{\rm reg}_{F+m^2} &= (-1)^{(d+1)/2}(4\pi)^{-d/2}\frac{\pi}{\Gamma(d/2)} B_{F+m^2}(d/2-1) \\
&\overset{m\to\infty}{\sim} (-1)^{(d+1)/2}(4\pi)^{-d/2}\pi\sum_{k\ge 0}\frac{m^{d-2-2k}}{k!\Gamma(d/2-k)} A_k(F)
\end{aligned}
\tag{37}
$$

and in even dimensions ($d = 2, 4, 6, \ldots$)

$$
\begin{aligned}
{\rm Tr}_{L^2}\, G^{\rm reg}_{F+m^2} =\ & (-1)^{d/2-1}\frac{(4\pi)^{-d/2}}{\Gamma(d/2)}\Big\{ B'_{F+m^2}(d/2-1) \\
& - \Big[\log\,\mu^2 + \Psi(d/2) - \mathcal{C}\Big]\, A_{d/2-1}(F+m^2)\Big\} \\
\overset{m\to\infty}{\sim}\ & (-1)^{d/2-1}(4\pi)^{-d/2}\Big\{\sum_{k=0}^{d/2-1}\frac{m^{d-2-2k}}{k!\Gamma(d/2-k)}[\mathcal{C} - \Psi(d/2-k) + \log\,\frac{m^2}{\mu^2}]A_k(F) \\
& + \sum_{k\ge d/2}\frac{(-1)^{k-d/2}}{k!} m^{d-2-2k}\Gamma(k+1-d/2)A_k(F)\Big\},
\end{aligned}
\tag{38}
$$

where

$$
B_{F+m^2}(q) = {\rm Tr}_{L^2} b_{F+m^2}(q) = \int\limits_M d{\rm vol}\,(x){\rm tr}_V\, b_{F+m^2}(x,x),
\tag{39}
$$

and $\mathcal{C} = -\Psi(1) = 0.577\ldots$ is the Euler's constant.

Thus, we see that

i) all the singularities of the Green function and the non-analytical parts thereof (proportional to $\sqrt{\sigma}$ in odd dimensions and to $\log\sigma$ in even dimensions) are determined by the HMDS-coefficients $a_k(F)$;

ii) there are no power singularities, i.e. $G^{\rm sing}_{F+m^2}=0$, in lower dimensions $d=1,2$;

iii) there is no logarithmic singularity (more generally, no logarithmic part at all) in odd dimensions;

iv) the regular part depends on the values of the function $b_{F+m^2}(q)$ at half-integer points and its derivative at integer points and is a global object that cannot be reduced to purely local information like the HMDS-coefficients.

The logarithmic part of the Green function is very important. On the one hand it determines, as usual, the renormalization properties of the regular part of the Green function, i.e. the derivative $\mu(\partial/\partial\mu)G^{\rm reg}_{F+m^2}$. In particular,

$$\mu\frac{\partial}{\partial\mu}{\rm Tr}_{\,L^2}G^{\rm reg}_{F+m^2}=\begin{cases}0 & \text{for odd } d\\ \dfrac{(4\pi)^{-d/2}}{\Gamma(d/2)}A_{d/2-1}(F+m^2) & \text{for even } d.\end{cases} \tag{40}$$

On the other hand, it is of crucial importance in studying the Huygens principle. Namely, the absence of the logarithmic part of the Green function is a necessary and sufficient condition for the validity of the Huygens principle for hyperbolic operators [17, 18, 19]. The HMDS-coefficients and, therefore, the logarithmic part of the Green function are defined for the hyperbolic operators just by analytic continuation from the elliptic case. Thus, the condition of the validity of Huygens principle reads

$$\sum_{k=0}^\infty\frac{\Gamma(d/2)}{k!\Gamma(k+d/2)}\left(\frac{\sigma}{2}\right)^k a_{k-1+d/2}(F+m^2)=0, \tag{41}$$

or, by using (21),

$$\sum_{k=0}^\infty\sum_{n=0}^{k-1+d/2}\frac{\Gamma(d/2)}{k!n!\Gamma(k-n+d/2)}\left(\frac{\sigma}{2}\right)^k m^{2(k-n)}a_n(F)=0, \tag{42}$$

By expanding this equation in covariant Taylor series using the methods of [7] one can obtain an infinite set of local conditions for validity of the Huygens principle.

$$\sum_{k=0}^{[n/2]}\frac{n!\Gamma(d/2)}{4^k k!(n-2k)!\Gamma(k+d/2)}(\vee^k g)\vee < n-2k|a_{k-1+d/2}(F+m^2) >=0, \tag{43}$$

where $\vee$ is the exterior symmetric tensor product, g is the metric tensor on the tangent bundle and $<n|a_k>$ denotes the diagonal value of the symmetrized covariant derivative of n-th order [7]. More explicitly,

$$[a_{d/2-1}(F+m^2)]=0, \tag{44}$$

$$[\nabla_\mu a_{d/2-1}(F+m^2)]=0, \tag{45}$$

$$[\nabla_{(\mu}\nabla_{\nu)}a_{d/2-1}(F+m^2)]+\frac{1}{2d}g_{\mu\nu}[a_{d/2}(F+m^2)]=0, \tag{46}$$

$$\dots$$

where the square brackets denote the diagonal value of two-point quantities: $[f(x,x')]=f(x,x)$.

III HIGHER ORDER OPERATORS

Let us now describe in detail the class of higher order operators we are going to study. We consider again a Riemannian manifold (M, g) and a vector bundle $V(M)$ with a connection ∇. Let $\mathcal{A}_{(i)}$, $(i = 1, \dots, N)$ be a set of different smooth sections of the vector bundle $T^*M \otimes \mathrm{End}(V)$. They define a set of *different* connections

$$\nabla_{(i)} = \nabla + \mathcal{A}_{(i)} \tag{47}$$

on the vector bundle $V(M)$, and therefore, a set of different generalized Laplacians

$$\begin{aligned} \Box_i &= \mathrm{tr}_g \nabla_{(i)} \nabla_{(i)} = g^{\mu\nu}(\nabla_\mu + \mathcal{A}_{(i)\,\mu})(\nabla_\nu + \mathcal{A}_{(i)\,\nu}) \\ &= \Box + g^{\mu\nu}(\mathcal{A}_{(i)\mu}\nabla_\nu + \nabla_\nu \mathcal{A}_{(i)\mu}) + g^{\mu\nu}\mathcal{A}_{(i)\mu}\mathcal{A}_{(i)\nu}. \end{aligned} \tag{48}$$

Let, further, Q_i be a set of different smooth sections of the vector bundle $\mathrm{End}(V)$. Then, we define a set of *different* Laplace type operators

$$F_i = -\Box_i + Q_i \tag{49}$$

and a higher order operator of a special form

$$H = (F_N + m^2) \cdots (F_2 + m^2)(F_1 + m^2). \tag{50}$$

III.1 Algebraical reduction to Laplace type operators

We are going to study the Green function $G_{H-\lambda}(x, x')$ of the higher order operator $H - \lambda$, where λ is an arbitrary sufficiently

large negative constant. To do this, we show, first, that the Green operator $G_{H-\lambda} = (H - \lambda)^{-1}$ can be reduced to the Green operator of an auxiliary Laplace type operator. This will mean that the Green function of the higher order differential operator will be reduced to the Green function of a *second-order* differential operator of Laplace type, which is described in previous section.

Theorem 1 *Let F_i, $(i = 1, \dots, N)$, be some positive elliptic operators and H be an operator defined by*

$$H = (F_N + m^2) \cdots (F_2 + m^2)(F_1 + m^2), \tag{51}$$

where m is a sufficiently large constant. Let I_{N-1} be the unit $(N-1) \times (N-1)$ matrix and O_{N-1} be the square $(N-1) \times (N-1)$ null matrix. Let P and Π be square $N \times N$ matrices defined by the following block form

$$P = \begin{pmatrix} 0 & I_{N-1} \\ \lambda & 0 \end{pmatrix}, \qquad \Pi = \frac{\partial}{\partial \lambda} P = \begin{pmatrix} 0 & O_{N-1} \\ 1 & 0 \end{pmatrix}, \tag{52}$$

where λ is a constant, and let $\widetilde{F} = F_1 \oplus \cdots \oplus F_N$ be a $N \times N$ diagonal matrix formed by the operators F_i

$$\widetilde{F} = \begin{pmatrix} F_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & F_N \end{pmatrix} \tag{53}$$

Let $\mathcal{F}$ be an operator defined by

$$\mathcal{F} = \widetilde{F} - P. \tag{54}$$

Let $G_{H-\lambda}$ and $G_{\mathcal{F}+m^2}$ be the Green operators of the operators $(H-\lambda)$ and $(\mathcal{F}+m^2)$ satisfying the equations

$$(H-\lambda)G_{H-\lambda} = 1 \tag{55}$$

and

$$(\mathcal{F}+m^2)G_{\mathcal{F}+m^2} = 1. \tag{56}$$

Then there holds

$$G_{H-\lambda} = \operatorname{tr}\Pi\, G_{\mathcal{F}+m^2}, \tag{57}$$

$$\frac{\partial}{\partial m^2} G_{H-\lambda} = -\frac{\partial}{\partial\lambda}\operatorname{tr} G_{\mathcal{F}+m^2}, \tag{58}$$

Proof. Let us introduce some auxiliary operators

$$\begin{aligned} Z_1 &= G_{H-\lambda} \\ Z_2 &= (F_1+m^2)Z_1 \\ &\vdots \\ Z_N &= (F_{N-1}+m^2)Z_{N-1}. \end{aligned} \tag{59}$$

Then the equation (55) can be rewritten as

$$(F_N+m^2)Z_N - \lambda G_{H-\lambda} = 1. \tag{60}$$

Let us collect the operators Z_i in a column-vector

$$Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_N \end{pmatrix} \tag{61}$$

The Green operator $G_{H-\lambda}$ is the first component of the vector Z and can be obtained by multiplying with the row-vector $\Pi_1^\dagger = (1, 0, \dots, 0)$

$$G_{H-\lambda} = \Pi_1^\dagger Z. \tag{62}$$

Further, let us define another column-vector

$$\Pi_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{63}$$

so that

$$\Pi = \Pi_N \otimes \Pi_1^\dagger. \tag{64}$$

It is not difficlut to show that the equations (59) can be rewritten in a matrix form

$$(\mathcal{F} + m^2) Z = \Pi_N. \tag{65}$$

Using the Green operator $G_{\mathcal{F}+m^2}$ of the operator $(\mathcal{F} + m^2)$ we obtain

$$Z = G_{\mathcal{F}+m^2} \Pi_N. \tag{66}$$

Finally, from (62) by taking into account (64) we get the Green operator $G_{H-\lambda}$

$$G_{H-\lambda} = \Pi_1^\dagger \, G_{\mathcal{F}+m^2} \Pi_N = \mathrm{tr}\, \Pi \, G_{\mathcal{F}+m^2}, \tag{67}$$

which proves the eq. (57). To prove eq. (58) we note that

$$\begin{aligned} \frac{\partial}{\partial\lambda} \mathrm{tr}\, G_{\mathcal{F}+m^2} &= -\mathrm{tr}\ G_{\mathcal{F}+m^2} \frac{\partial \mathcal{F}}{\partial\lambda} G_{\mathcal{F}+m^2} \\ &= -\mathrm{tr}\ \Pi \, G^2_{\mathcal{F}+m^2}. \end{aligned} \tag{68}$$

Further, from the definition of the Green operator we have, obviosly,

$$\frac{\partial}{\partial m^2} \mathrm{tr}\ \Pi \, G_{\mathcal{F}+m^2} = -\mathrm{tr}\ \Pi \, G^2_{\mathcal{F}+m^2}. \tag{69}$$

Comparing (68) and (69) and using the eq. (67) we convince ourselves that eq. (58) is correct too.

Let us make some remarks. First, 'tr' denotes here the usual matrix trace and has nothing to do with the trace over the bundle indices which are left intact. Second, the operator $\widetilde{F}$ has, obviously, the form

$$\widetilde{F} = -\widetilde{\Box} + \widetilde{Q}, \tag{70}$$

where $\widetilde{\Box} = \Box_1 \oplus \cdots \oplus \Box_N$ and $\widetilde{Q} = Q_1 \oplus \cdots \oplus Q_N$ have the same diagonal structure. Obviously

$$\widetilde{\Box} = \mathrm{tr}_g \widetilde{\nabla}\widetilde{\nabla} \tag{71}$$

where $\widetilde{\nabla} = \nabla_1 \oplus \cdots \oplus \nabla_N$ is a new connection which also has the same diagonal structure. Therefore, the operator $\mathcal{F}$ reads

$$\mathcal{F} = -\widetilde{\Box} + \widetilde{Q} - P. \tag{72}$$

Remember that the matrix P is constant. Thus, both the operator $\widetilde{F}$ and $\mathcal{F}$ are *second-order* elliptic Laplace type differential operators. However, the operator $\mathcal{F}$ is *not self-adjoint* because the matrix P is not symmetric. In spite of this, the operator $(\mathcal{F} + m^2)$ for sufficiently large m is non-degenerate, i.e. there exists a well defined Green operator $G_{\mathcal{F}+m^2}$, which justifies made assumptions.

III.1.1 Properties of the matrix P

The matrix P plays the role of the matrix term and is of great importance in further considerations. That is why, we state below some important properties of it.

First of all, we note that the matrix Π is nilpotent

$$\Pi^2 = 0 \tag{73}$$

and, of course, degenerate. The matrix P is not degenerate. The matrix P as well as its powers belong to the class of two-diagonal matrices. We compute first the powers of this matrix.

$$P = \begin{pmatrix} 0 & I_{N-1} \\ \lambda & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0 & I_{N-2} \\ \lambda I_2 & 0 \end{pmatrix}, \ \dots, \ P^{N-1} = \begin{pmatrix} 0 & 1 \\ \lambda I_{N-1} & 0 \end{pmatrix}, \tag{74}$$

$$P^N = \lambda I_N. \tag{75}$$

It is also not difficult to show that

$$\det\,(z - P) = z^N - \lambda. \tag{76}$$

Therefore, the eigenvalues p_j of the matrix P are just the N-th roots of λ

$$p_j = (-\lambda)^{1/N} e^{i\,\pi(2j-1)/N}, \qquad j = 1, \dots, N, \tag{77}$$

where $|\arg(-\lambda)| < \pi$. The roots p_j lie on the circle $|z| = |\lambda|^{1/N}$.

The higher powers of the matrix P are expressed in terms of the lower order powers (74) and (75)

$$P^{iN+j} = \lambda^i P^j, \tag{78}$$

where $j = 0, 1, 2, \dots, N-1$ and $i = 0, 1, 2, \dots$. Using these formulas we find that for any function $f(z)$ that is analytic at the points $p_j(\lambda)$ there holds

$$\begin{aligned} f(P) &= \frac{1}{2\pi i} \int\limits_{C_P} dz\, f(z)\,(z - P)^{-1} \\ &= \sum_{j=0}^{N-1} \beta_{N-1-j}(\lambda; f) P^j \end{aligned} \tag{79}$$

where

$$\beta_n(\lambda; f) = \frac{1}{2\pi i} \int\limits_{C_P} dz\, \frac{z^n}{z^N - \lambda} f(z) \tag{80}$$

and the contour C_P goes counter-clockwise around the points p_j in such a way that it does not contain the singularities of the function $f(z)$ — inside the contour C_P there must be no other singularities except for the points p_j. Evidently, the coefficients $\beta_n(\lambda; f)$ possess the following 'periodicity' property

$$\beta_{n+kN}(\lambda; f) = \lambda^k \beta_n(\lambda; f), \qquad k = 0, 1, 2, \dots, \tag{81}$$

so that there are essentially only N independent coefficients $\beta_0, \beta_1, \dots, \beta_{N-1}$.

The eqs. (79) and (85) determine any function of the matrix P. In particular, using the traces of the powers of the matrix P

$$\mathrm{tr}\ P^n = \begin{cases} N\lambda^i & \text{for } n = iN,\ i = 0, 1, 2, \dots, \\ 0 & \text{otherwise} \end{cases} \tag{82}$$

we can compute the trace of any function of P

$$\mathrm{tr}\ f(P) = N\beta_{N-1}(\lambda; f). \tag{83}$$

More generally,

$$\mathrm{tr}\ P^n f(P) = N\beta_{N-1+n}(\lambda; f). \tag{84}$$

Herefrom we obtain the coefficients $\beta_n(\lambda; f)$

$$\begin{aligned} \beta_n(\lambda; f) &= \frac{1}{N\lambda}\sum_{j=1}^{N} p_j^{n+1} f(p_j) \\ &= -\frac{1}{N}(-\lambda)^{(n-N+1)/N}\sum_{j=1}^{N} e^{i\pi(2j-1)(n+1)/N} f(p_j) \end{aligned} \tag{85}$$

From (82), it follows, in particular,

$$\sum_{j=1}^{N} p_j^n = 0 \qquad \text{for } n \neq 0, \pm N, \pm 2N, \dots. \tag{86}$$

Moreover, using the definition (52) we calculate the traces of the product of the powers of the matrix P with the matrix Π

$$\mathrm{tr}\ \Pi\, P^n = \frac{1}{(n+1)}\frac{\partial}{\partial\lambda}\mathrm{tr}\ P^{n+1} = \begin{cases} \lambda^i & \text{for } n = iN + N - 1,\ i = 0, 1, 2, \dots, \\ 0 & \text{otherwise} \end{cases}. \tag{87}$$

By using the equation

$$\frac{\partial}{\partial\lambda}\mathrm{tr}\ f(P) = \mathrm{tr}\ \frac{\partial P}{\partial\lambda} f'(P) \tag{88}$$

where $f'(z) = (\partial/\partial z) f(z)$, and the eq. (52) or using directly the eq. (87) we calculate the trace of the product of the function $f(P)$ with the matrix Π

$$\mathrm{tr}\ \Pi\, f(P) = \beta_0(\lambda; f) = -\frac{1}{N}(-\lambda)^{-(N-1)/N}\sum_{j=1}^{N} e^{i\pi(2j-1)/N} f(p_j). \tag{89}$$

More generally, one can obtain also

$$\mathrm{tr}\ \Pi(z-P)^{-1} = -\frac{\partial}{\partial\lambda}\log\ \det(z-P) = \frac{1}{z^N-\lambda} \tag{90}$$

as well as

$$\mathrm{tr}\ \Pi\, e^{tP} = \frac{1}{2\pi i}\int\limits_{C_P} dz \frac{e^{tz}}{z^N-\lambda} = -\frac{1}{N}(-\lambda)^{-(N-1)/N}\sum_{j=1}^{N} e^{i\,\pi(2j-1)/N} e^{tp_j}. \tag{91}$$

In the next section we will need also the following lemma, which is actually a generalization of the eq. (82).

Lemma 1 *Let P and Π be the $N\times N$ matrices defined by (52) and $B_1,\dots,B_{N-2}$ be arbitrary diagonal matrices. Then for any $k\le N-2$ and arbitrary non-negative integers $n_1,\dots,n_k$ satisfying the condition $n_1+\cdots+n_k\le N-2$ there holds*

$$\mathrm{tr}\ \Pi\, B_1P^{n_1}B_2P^{n_2}\cdots B_kP^{n_k} = 0. \tag{92}$$

III.1.2 Properties of the symbol $\sigma(\mathcal{F})$

Let us consider now the symbol $\sigma(\mathcal{F})$ of the operator $\mathcal{F}$. It has the form

$$\sigma(\mathcal{F}) = \sigma(\widetilde{F}) - P, \tag{93}$$

where $\sigma(\widetilde{F})$ is a diagonal matrix.

That is why let us consider the following matrix

$$M \equiv \tilde{\mu} - P, \tag{94}$$

where $\tilde{\mu} = \mu_1\oplus\cdots\oplus\mu_N$ is a diagonal matrix with some constants μ_j on the diagonal. Note that the matrices $\tilde{\mu}$ and P do not commute with each other $[\tilde{\mu}, P]\neq 0$.

We are interested in the eigenvalues q_j of the matrix M. It is not difficult to calculate

$$\mathrm{tr}\, M^n = \mathrm{tr}\,\tilde{\mu}^n = \sum_{j=1}^{N}\mu_j^n, \qquad (n=1,2,\dots,N-1) \tag{95}$$

$$\mathrm{tr}\, M^N = \sum_{j=1}^{N}\mu_j^N + (-1)^N N\lambda, \tag{96}$$

$$\det\ (z-M) = \prod_{j=1}^{N}(z-\mu_j) - (-1)^N\lambda. \tag{97}$$

Thus the eigenvalues q_j of the matrix M are determined either from one algebraical equation of N-th order

$$(\mu_1-z)\cdots(\mu_N-z)-\lambda = 0 \tag{98}$$

or from the system of N equations

$$q_1 + \cdots + q_N = \sum_{j=1}^{N} \mu_j \tag{99}$$

$$\cdots$$

$$q_1^{N-1} + \cdots + q_N^{N-1} = \sum_{j=1}^{N} \mu_j^{N-1} \tag{100}$$

$$q_1^{N} + \cdots + q_N^{N} = \sum_{j=1}^{N} \mu_j^{N} + (-1)^N N\lambda. \tag{101}$$

This enables one to calculate the traces

$$\mathrm{tr}\ f(M) = \sum_{j=1}^{N} f(q_j). \tag{102}$$

One can get also analogous to (80) integral representation

$$\mathrm{tr}\ f(M) = \frac{1}{2\pi i}\int\limits_{C_M} dz\, f(z)\frac{A(z,\mu,\lambda)}{(z-\mu_1)\cdots(z-\mu_N)-(-1)^N\lambda} \tag{103}$$

where C_M is a counter-clockwise contour around the spectrum of the matrix M and

$$A(z,\mu,\lambda) = \sum_{j=1}^{N}\prod_{k=1;\,k\neq j}^{N}(z-q_k) \tag{104}$$

is a polynomial of order $(N-1)$.

Further, by using the equation

$$\frac{\partial}{\partial\lambda}\mathrm{tr}\ f(M) = \mathrm{tr}\ \frac{\partial M}{\partial\lambda}\, f'(M), \tag{105}$$

where $f'(z) = (\partial/\partial z) f(z)$, and the obvious equation

$$\frac{\partial}{\partial\lambda}M = -\Pi, \tag{106}$$

which follows from the definition of the matrix M and the eq. (52), we calculate the trace of the product of the function $f'(M)$ with the matrix Π

$$\mathrm{tr}\ \Pi\, f'(M) = -\frac{\partial}{\partial\lambda}\mathrm{tr}\ f(M). \tag{107}$$

Using the eqs. (95) and (96) we get, in particular

$$\mathrm{tr}\ \Pi\, M^n = -\frac{1}{n+1}\frac{\partial}{\partial\lambda}\mathrm{tr}\ M^{n+1} = 0 \qquad \text{for } n = 0,1,\ldots,N-2 \tag{108}$$

$$\mathrm{tr}\ \Pi\, M^{N-1} = -\frac{1}{N}\frac{\partial}{\partial\lambda}\mathrm{tr}\ M^{N} = (-1)^{N-1}. \tag{109}$$

Note that (108) is a particular case of the eq. (92). Further, from (99) – (101) one obtains

$$\sum_{j=1}^{N}\frac{\partial q_j}{\partial\lambda}q_j^n = 0 \qquad \text{for } n=0,1,2,\dots,N-2. \tag{110}$$

$$\sum_{j=1}^{N}\frac{\partial q_j}{\partial\lambda}q_j^{N-1} = (-1)^N. \tag{111}$$

Further, we get also

$$\mathrm{tr}\,\Pi\,(z-M)^{-1} = \frac{\partial}{\partial\lambda}\log\,\det\,(z-M) = \frac{(-1)^{N-1}}{(z-\mu_1)\cdots(z-\mu_N)-(-1)^N\lambda}, \tag{112}$$

$$\mathrm{tr}\,\Pi\,M^{-k} = \frac{(-1)^N}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}\frac{1}{(z-\mu_1)\cdots(z-\mu_N)-(-1)^N\lambda}\Bigg|_{z=0}, \tag{113}$$

$$\mathrm{tr}\,\Pi\,M^{k} = -\frac{1}{\Gamma(k-N+2)}\left(\frac{\partial}{\partial y}\right)^{k-N+1}\frac{1}{(y\mu_1-1)\cdots(y\mu_N-1)-y^N\lambda}\Bigg|_{y=0} \tag{114}$$

Note that for $k\le N-2$ the last equation gives zero in accordance with (109).

This leads to a very simple and convenient representation

$$\mathrm{tr}\,\Pi\,f(M) = (-1)^{N-1}\frac{1}{2\pi i}\int\limits_{C_M} dz\,\frac{f(z)}{(z-\mu_1)\cdots(z-\mu_N)-(-1)^N\lambda} \tag{115}$$

We will need also, in particular, the following formula

$$\begin{aligned}\mathrm{tr}\,\Pi\,e^{-tM} &= (-1)^{N-1}\frac{1}{2\pi i}\int\limits_{C_M} dz\,\frac{e^{-tz}}{(z-\mu_1)\cdots(z-\mu_N)-(-1)^N\lambda}\\ &= \frac{1}{t}\frac{\partial}{\partial\lambda}\mathrm{tr}\;e^{-tM} = -\sum_{j=1}^{N}\frac{\partial q_j}{\partial\lambda}e^{-tq_j}.\end{aligned} \tag{116}$$

From (108) it follows that this function is of order $O(t^{N-1})$ as $t\to0$. Using eq. (98) we obtain

$$\frac{\partial q_j}{\partial\lambda} = -\left\{\sum_{k=1}^{N}\prod_{i=1;\,i\ne k}^{N}(\mu_i-q_j)\right\}^{-1}. \tag{117}$$

In particular case $\lambda=0$ we have

$$q_j\Big|_{\lambda=0} = \mu_j, \tag{118}$$

hence,

$$\frac{\partial q_j}{\partial\lambda}\Big|_{\lambda=0} = -\prod_{k=1;\,k\ne j}^{N}\frac{1}{(\mu_k-\mu_j)} \tag{119}$$

and

$$\mathrm{tr}\,\Pi\,e^{-tM}\Big|_{\lambda=0} = \sum_{j=1}^{N}e^{-t\mu_j}\prod_{k=1;\,k\ne j}^{N}\frac{1}{(\mu_k-\mu_j)}. \tag{120}$$

These formulas will be useful when studying the heat kernel for the operator $\mathcal{F}$.

III.2 Green function of the operator $H - \lambda$

This shows that the problem is reduced to a Laplace type operator $\mathcal{F}$ with an additional matrix structure. Using the usual heat kernel representation for the Green function $\mathcal{G}$ of the Laplace type operator $\mathcal{F}$ we have from eq. (57)

Corollary 1

$$G_{H-\lambda} = \int_0^\infty dt\, e^{-tm^2} \mathrm{tr}\,\Pi\, \exp(-t\mathcal{F}), \tag{121}$$

or in the kernel form

$$G_{H-\lambda} = (4\pi)^{-d/2}\Delta^{1/2} \int_0^\infty dt\, t^{-d/2} \exp\left(-tm^2 - \frac{\sigma}{2t}\right) \omega_{\mathcal{F}}(t), \tag{122}$$

where

$$\omega_{\mathcal{F}}(t) = \mathrm{tr}\,\Pi\,\Omega_{\mathcal{F}}(t), \tag{123}$$

$\Omega_{\mathcal{F}}(t) = \Omega_{\mathcal{F}}(t|x, x')$ *is the transfer function of the operator* $\mathcal{F}$ *(54).*

Thus we see that there is analogous heat kernel representation for the Green function of higher-order operator $H - \lambda$ with a new ‘transfer function’ $\omega_{\mathcal{F}}(t)$.

III.2.1 Power of a Laplace type operator

The simplest case is, of course, the case of equal potential terms Q_j and the connections $\nabla^{(i)}$, i.e.

$$Q_i = Q, \qquad \nabla^{(i)} = \nabla. \tag{124}$$

In this case all operators F_j are equal to each other

$$F_j = F = -\Box + Q, \tag{125}$$

and the operator H is the N-th power of the Laplace type operator $F + m^2$

$$H = (F + m^2)^N. \tag{126}$$

Further, in this simple case we have

$$\mathcal{F} = F - P \tag{127}$$

and the operator F commutes with the matrix P

$$[F, P] = 0. \tag{128}$$

Therefore,

$$\exp(-t\mathcal{F}) = \exp(tP)\exp(-tF). \tag{129}$$

and, using eqs. (57) and (26) we get

$$\omega_{\mathcal{F}}(t) = f(t)\Omega_F(t), \qquad f(t) = \mathrm{tr}\,\Pi\exp(tP), \tag{130}$$

which is determined by (91) and (85)

$$f(t) = \frac{1}{2\pi i}\int\limits_{C_P} dz \frac{e^{tz}}{z^N - \lambda} = -\frac{1}{N}(-\lambda)^{-(N-1)/N}\sum_{j=1}^{N} e^{i\pi(2j-1)/N} e^{tp_j(\lambda)}. \qquad (131)$$

On the other hand, expanding the integrand in powers of λ and using the formula (163) we obtain $f(t)$ in form of a power series

$$f(t) = t^{N-1}\sum_{k\geq 0}\frac{\lambda^k t^{Nk}}{(Nk+N-1)!}. \qquad (132)$$

Thus, the function $\omega_{\mathcal{F}}(t)$ is of order $O(t^{N-1})$ as $t \to 0$.

In case $\lambda = 0$ we obtain herefrom the usual heat kernel representation for the inverse power of the operator $(F + m^2)$

$$G_H = (F + m^2)^{-N} = \frac{1}{(N-1)!}\int\limits_0^\infty dt\, t^{N-1} e^{-tm^2} \exp(-tF). \qquad (133)$$

III.2.2 Constant perturbations of the potential terms

Let us consider now the case when the Laplace type operators F_j differ from each other only by a constant part of the potential terms, i.e. the connection $\nabla^{(j)}$ is the same, but the endomorphisms Q_j are different

$$Q_j = Q + \mu_j, \qquad (134)$$

so that

$$F_i = F + \mu_j, \qquad (135)$$

where

$$F = -\Box + Q. \qquad (136)$$

The operator H has the form

$$H = (F + m^2 + \mu_N)\cdots(F + m^2 + \mu_2)(F + m^2 + \mu_1). \qquad (137)$$

The operators F_i, even if different, commute. The corresponding operator $\mathcal{F}$ has the form

$$\mathcal{F} = F + M, \qquad (138)$$

where $M = \tilde{\mu} - P$ and $\tilde{\mu} = \mu_1 \oplus \cdots \oplus \mu_N$ is a diagonal matrix with the constants μ_j on the diagonal. The matrix M commutes with the operator F

$$[F, M] = 0. \qquad (139)$$

Therefore,

$$\omega_{\mathcal{F}}(t) = h(t)\Omega_F(t), \qquad h(t) = \mathrm{tr}\ \Pi \exp(-tM). \qquad (140)$$

Using the eigenvalues of the matrix M, which are determined by the eq. (98), we obtained this function in the form

$$\begin{aligned} h(t) &= (-1)^{N-1}\frac{1}{2\pi i}\int\limits_{C_M} dz\, \frac{e^{-tz}}{(z-\mu_1)\cdots(z-\mu_N)-(-1)^N\lambda} \\ &= -\sum_{j=1}^{N}\frac{\partial q_j}{\partial\lambda}e^{-tq_j}. \end{aligned} \tag{141}$$

As it has been shown in section 3.2, this function is of order $O(t^{N-1})$ as $t\to 0$. For $\lambda=0$ we obtained

$$h(t)\Big|_{\lambda=0} = \sum_{j=1}^{N}\left[\prod_{k=1;\,k\neq j}^{N}\frac{1}{(\mu_k-\mu_j)}\right]e^{-t\mu_j}. \tag{142}$$

III.2.3 General case

As we have seen, in particular cases studied before the function $\omega_{\mathcal{F}}(t)$ defined by (123) is of order $O(t^{N-1})$ as $t\to 0$. It is almost clear that this is valid also in general, since the main contribution to the asymptotics give the constant background fields. This can be formulated in form of a lemma.

Lemma 2 *Let $\mathcal{F}$ be the Laplace type operator defined by (54) and Π be the matrix defined by (52). Then for its HMDS-coefficients $a_k(\mathcal{F})$ there holds*

$$\mathrm{tr}\,\Pi\, a_k(\mathcal{F}) = 0 \qquad \text{for } k=0,1,\dots,N-2. \tag{143}$$

In other words, for the asymptotic expansion as $t\to 0$ of its transfer function $\omega_{\mathcal{F}}(t)$ there holds

$$\omega_{\mathcal{F}}(t)\Big|_{t\to 0} \sim \frac{(-t)^{N-1}}{(N-1)!}\sum_{k\geq 0}\frac{(-t)^k}{k!}c_k(\mathcal{F}), \tag{144}$$

where

$$c_k(\mathcal{F}) = \frac{(N-1)!k!}{(N+k-1)!}\mathrm{tr}\ \Pi\, a_{k+N-1}(\mathcal{F}). \tag{145}$$

Proof. The coefficients a_k can be calculated in form of a covariant Taylor series near the diagonal (for details and notation see [7])

$$a_k = \sum_{n\geq 0}|n><n|a_k>, \tag{146}$$

where $|n>$ is the covariant Taylor basis and $<n|a_k>$ are the diagonal values of the symmetrized covariant derivatives. $<n|a_k>$ have the following general form [7]

$$<n|a_k> = \sum_{n_1,\dots,n_{k-1}\geq 0} c(n_1,\dots,n_{k-1}) <n|M|n_{k-1}><n_{k-1}|M|n_{k-2}>\cdots<n_1|M|0>, \tag{147}$$

where $c(n_1, \dots, n_{k-1})$ are some constants and $< m|M|n >$ are some End (V)-valued tensors that vanish for $n > m + 2$ and for $n = m + 1$. Important for us is that, since the matrix P is constant, it appears only in $< n|M|n >$

$$< n|M|n >= P + B, \tag{148}$$

where B is a diagonal matrix. *All* others tensors $< n|M|m >$ do not depend on the matrix P and are diagonal—they are actually polynomials in the curvatures and their covariant derivatives [7]. So, all the terms in the sum (147) have the form

$$B_1 P^{n_1} \dots B_j P^{n_j}, \qquad j = 1, 2, \dots, k, \tag{149}$$

where B_i are some diagonal matrices and the total power of the matrix P is restricted by $n_1 + \cdots + n_j \le k$. Therefore, from (92) we find that $\operatorname{tr}\Pi < n|a_k >= \operatorname{tr}\,\Pi\, a_k = 0$ for $k \le N - 2$, which proves the lemma.

Actually, using (148) and (87) one can prove more, namely,

$$c_0(\mathcal{F}) = \operatorname{tr}\,\Pi\, a_{N-1}(\mathcal{F}) = \operatorname{tr}\Pi\, P^{N-1} = 1. \tag{150}$$

III.2.4 Hadamard expansion of the Green function of the operator $H - \lambda$

The behavior of the Green function of Laplace type operators near diagonal is studied in the previous section. So, by using the eqs. of sect. 2 we obtain from (57) the Hadamard series for the Green function of the higher order operator $H - \lambda$

$$G_{H-\lambda} = G^{\rm sing}_{H-\lambda} + G^{\rm non-anal}_{H-\lambda} + G^{\rm reg}_{H-\lambda}, \tag{151}$$

where the singular part reads

$$G^{\rm sing}_{H-\lambda} = (4\pi)^{-\frac{d}{2}} \frac{\Delta^{1/2}}{(N-1)!} \sum_{k=0}^{\left[\frac{(d+1)}{2}\right]-N-1} (-1)^{N+k} \frac{\Gamma\left(\frac{d}{2} - k - N\right)}{k!} \left(\frac{2}{\sigma}\right)^{\frac{d}{2}-k-N} c_k(\mathcal{F} + m^2). \tag{152}$$

Further, we get in odd dimensions:

$$G^{\rm non-anal}_{H-\lambda} \sim (-1)^{\frac{d-1}{2}} (4\pi)^{-\frac{d}{2}} \frac{\Delta^{\frac{1}{2}}}{(N-1)!} \sum_{k=0}^{\infty} \frac{\pi\, (\sigma/2)^{k+1/2}\, c_{k-N+(d+1)/2}(\mathcal{F} + m^2)}{\Gamma\left(k - N + 1 + \frac{d+1}{2}\right)\Gamma\left(k + \frac{3}{2}\right)} \tag{153}$$

$$G^{\rm reg}_{H-\lambda} \sim (-1)^{\frac{(d+1)}{2}} (4\pi)^{-\frac{d}{2}} \frac{\Delta^{1/2}}{(N-1)!} \sum_{k=0}^{\infty} \frac{\pi\, (\sigma/2)^k}{k!\Gamma(k - N + 1 + d/2)} \varphi_{\mathcal{F}+m^2}(k - N + d/2), \tag{154}$$

and in even dimensions:

$$G^{\rm non-anal}_{H-\lambda} \sim (-1)^{d/2-1} (4\pi)^{-d/2} \frac{\Delta^{1/2}}{(N-1)!} \log\left(\frac{\mu^2\sigma}{2}\right) \sum_{k=0}^{\infty} \frac{(\sigma/2)^k\, c_{k-N+d/2}(\mathcal{F} + m^2)}{k!\Gamma(k - N + 1 + \frac{d}{2})} \tag{155}$$

$$
\begin{aligned}
G^{\rm reg}_{H-\lambda} \quad &\sim \quad (-1)^{d/2-1}(4\pi)^{-d/2}\frac{\Delta^{1/2}}{(N-1)!}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k-N+1+d/2)}\left(\frac{\sigma}{2}\right)^k \\
&\times \quad \left\{\varphi'_{\mathcal{F}+m^2}(k-N+d/2) - \left[\log\mu^2 + \Psi(k+1) + \Psi(k+d/2)\right] c_{k-N+\frac{d}{2}}(\mathcal{F}+m^2)\right\}
\end{aligned}
\tag{156}
$$

where

$$
\varphi_{\mathcal{F}+m^2}(q) \equiv \frac{(N-1)!\Gamma(q+1)}{\Gamma(q+N)}\mathrm{tr}\ \Pi\, b_{\mathcal{F}+m^2}(q+N-1). \tag{157}
$$

Thus, we find that

i) the structure of the diagonal singularities of the Green function $G_{H-\lambda}$ is determined by the coeffcieints $c_k(\mathcal{F}+m^2)$;

ii) in odd dimenions there is no logarithmic singularity;

iii) for $N > d/2$ there are *no* singularities at all and the Green function is regular on the diagonal, i.e. there exists finite coincidence limit $G^{\rm diag}_{H-\lambda} \equiv G_{H-\lambda}(x,x) = G^{\rm reg}_{H-\lambda}(x,x)$;

iv) for $N = [(d+1)/2]$ there are no singularities for odd dimension $d = 2N-1$ and there is only logarithmic singularity in even dimension $d = 2N$.

v) The condition for the validity of the Huygens principle for the operator $H-\lambda$ in even dimensions reads

$$
\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k-N+1+d/2)}\left(\frac{\sigma}{2}\right)^k c_{k-N+d/2}(\mathcal{F}+m^2) = 0. \tag{158}
$$

Similarly to (43) this also produces an infinite set of local conditions.

III.3 Heat kernel of the operator H

The heat semigroup for the operator H can be obtained by using the standard formula

$$
\begin{aligned}
\exp(-tH) \quad &= \quad -\frac{1}{2\pi i}\int_{C_H} d\lambda e^{-t\lambda} G_{H-\lambda} \\
&= \quad \frac{t^{-k}}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\lambda e^{-t\lambda}\left(\frac{\partial}{\partial\lambda}\right)^k G_{H-\lambda},
\end{aligned}
\tag{159}
$$

where the contour of integration C_H goes counter-clockwise around the spectrum of the operator H, k is some sufficiently large integer and c is a negative constant. Since the operator H is a positive self-adjoint operator its spectrum lies on the positive real half-axis, i.e. the contour C_H should go from $i\varepsilon+\infty$ to $-i\varepsilon+\infty$ enclosing the origin. Using

the Theorem 1 one can express the heat semigroup for the higher order operator H in terms of the heat semigroup of the Laplace type operator $\mathcal{F}$. We have first

$$\exp(-tH) = \frac{t^{-k}}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\lambda\, e^{-t\lambda}\left(\frac{\partial}{\partial\lambda}\right)^k \mathrm{tr}\,\Pi\, G_{\mathcal{F}+m^2}, \tag{160}$$

Now, using the heat kernel representation of the Green operator $G_{\mathcal{F}+m^2}$ we obtain

Corollary 2 *The heat semigroup for the operator H is determined by the heat semigroup for the operator $\mathcal{F}$*

$$\exp(-tH) = \frac{t^{-k}}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\lambda\, e^{-t\lambda}\int\limits_0^\infty ds\, e^{-sm^2}\left(\frac{\partial}{\partial\lambda}\right)^k \mathrm{tr}\;\Pi\;\exp(-s\mathcal{F}) \tag{161}$$

and, consequently, the heat kernel for the operator H is given by

$$U_H(t) \;=\; (4\pi)^{-d/2}\Delta^{1/2}\frac{t^{-k}}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\lambda\, e^{-t\lambda}\int\limits_0^\infty ds\, s^{-d/2}\exp\left(-sm^2-\frac{\sigma}{2s}\right)\left(\frac{\partial}{\partial\lambda}\right)^k \omega_{\mathcal{F}}(s). \tag{162}$$

We studied the expansion of the Green function of a Laplace type operator in previous section in detail. The important fact is, that the HMDS-coefficients $a_k(\mathcal{F}+m^2)$ for the Laplace type operator $(\mathcal{F}+m^2)$ are polynomial in λ, they are, of course, also polynomial in m^2. This means that the singular part of the Green function (including the logarithmic singularity), which are expressed in terms of $a_k(\mathcal{F}+m^2)$, are also polynomial in λ. On the other hand, it is well known that

$$-\frac{1}{2\pi i}\int\limits_{C_H} d\lambda\, e^{-\lambda}(-\lambda)^z = \frac{1}{\Gamma(-z)}. \tag{163}$$

For integer $z = k = 0, 1, 2, \ldots$ we have, therefore,

$$-\frac{1}{2\pi i}\int\limits_{C_H} d\lambda\, e^{-\lambda}(-\lambda)^k = 0. \tag{164}$$

Thus, the singular part, which is polynomial in λ, does not contribute to the integral (160). Therefore, one can substitute in this equation instead of the whole Green function only the regular part of it $G^{\rm reg}$, go to the diagonal and take the trace

$$\mathrm{Tr}\,_{L^2}\exp(-tH) = \frac{t^{-k}}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\lambda\, e^{-t\lambda}\left(\frac{\partial}{\partial\lambda}\right)^k \mathrm{Tr}\,_{L^2}\mathrm{tr}\,\Pi\, G^{\rm reg}_{\mathcal{F}+m^2}, \tag{165}$$

Finally, using the eqs. (37) and (38) we have: in odd dimensions

$$\mathrm{Tr}\,_{L^2}\exp(-tH) = \frac{(-1)^{(d+1)/2}(4\pi)^{-d/2}\pi}{(N-1)!\Gamma(d/2-N+1)}\frac{t^{-k}}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\lambda\, e^{-t\lambda}\left(\frac{\partial}{\partial\lambda}\right)^k \Phi_{\mathcal{F}+m^2}(d/2-N), \tag{166}$$

and in even dimensions

$$\mathrm{Tr}_{\,L^2}\exp(-tH) = \frac{(-1)^{d/2-1}(4\pi)^{-d/2}}{(N-1)!\Gamma(d/2-N+1)}\frac{t^{-k}}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\lambda\, e^{-t\lambda}\left(\frac{\partial}{\partial\lambda}\right)^k \Phi'_{\mathcal{F}+m^2}(d/2-N), \tag{167}$$

where $\Phi_{\mathcal{F}+m^2}(q) = \mathrm{Tr}_{\,L^2}\varphi_{\mathcal{F}+m^2}(q)$ and $\Phi'_{\mathcal{F}+m^2}(q) = (d/dq)\Phi_{\mathcal{F}+m^2}(q)$.

These formulas may be used, for example, to compute the asymptotic expansion of the functional trace of the heat kernel for the higher order operator H, which is known to have the following general form [3]

$$\mathrm{Tr}_{\,L^2}\exp(-tH) \sim (4\pi)^{-d/2}\frac{\Gamma\left[d/(2N)\right]}{N\Gamma\left(d/2\right)}\,t^{-d/(2N)}\sum_{k\geq 0} t^{k/N}E_k(H). \tag{168}$$

We normalize the coefficients $E_k(H)$ so that $E_0(H) = \mathrm{Tr}_{L^2}\,1$. For $N=1$ the coefficients E_k are determined by the HMDS-coefficients $E_k = [(-1)^k/k!]A_k$. However, one should note that to compute the coefficients $E_k(H)$ one needs to know the asymptotics of the function $\omega_{\mathcal{F}}(t)$ (or the functions $\Phi_{\mathcal{F}+m^2}(d/2-1)$ and $\Phi'_{\mathcal{F}+m^2}(d/2-1)$) as $\lambda \to -\infty$, which is unknown, in general, in terms of the known asymptotic expansion as $t \to 0$, i.e. in terms of the HMDS-coefficients for a Laplace type operator. The known asymptotics of these functions as $s \to 0$ and $m \to \infty$ do not contribute at all in the eqs. (162), (166) and (167).

ACKNOWLEDGMENTS

I am greatly indebted to Peter Gilkey and Rainer Schimming for stimulating and fruitful discussions. This work was supported by the Deutsche Forschungsgemeinschaft.

References

[1] J. Hadamard, *Lectures on Cauchy's Problem*, in: *Linear Partial Differential Equations* (Yale U. P., New Haven, 1923)

[2] B. S. De Witt, *Dynamical Theory of Groups and Fields*, (New York: Gordon and Breach, 1965)

[3] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah - Singer Index Theorem* (Publish or Perish, Wilmington, DE, USA, 1984)

[4] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. **C119**, 1 (1985)

[5] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the theory of quantized fields*, (Moscow: Nauka, 1976)

[6] I. G. Avramidi, *Covariant Methods for the Calculation of the Effective Action in Quantum Field Theory and Investigation of Higher-Derivative Quantum Gravity*, PhD thesis, Moscow State University (1986), UDK 530.12:531.51, [in Russian]; Transl.: hep-th/9510140

[7] I. G. Avramidi, Nucl. Phys. **B355**, 712 (1991)

[8] B. S. De Witt and R. W. Brehme, Ann. Physics **9**, 220 (1960)

[9] J. L. Synge, *Relativity: The general theory*, (Amsterdam: North-Holland, 1960)

[10] S. Minakshisundaram, J. Indian Math. Soc. **17**, 158 (1953)

[11] R. T. Seeley, "Complex powers of an elliptic operator", in: *Singular Integrals*, Proc. Symp. Pure Math. (Chicago, Ill, 1966), Vol. 10, (Amer. Math. Soc., Providence, RI, 1967), pp. 288-307

[12] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975)

[13] I. G. Avramidi, Phys. Lett. B **238**, 92 (1990)

[14] P. Amsterdamski, A. L. Berkin and D. J. O'Connor, Class. Quantum Grav. **6**, 1981 (1989)

[15] I. G. Avramidi and R. Schimming, *Algorithms for the calculation of the heat kernel coefficients*, in: *'Quantum Field Theory under the Influence of External Conditions'*, Ed. M. Bordag, Teubner-Texte zur Physik, Vol. 30, (Stuttgart: Teubner, 1996), p. 150

[16] R. Schimming, *Calculation of the heat kernel coefficients*, in: *Analysis, Geometry and Groups. A Riemann Legacy Volume*, Eds. H. M. Srivastava and Th. M. Rassias, (Hadronic Press, Palm Harbour, 1993),

p. 627

[17] R. Schimming, Beitr. z. Analysis, **11**, 45 (1978)

[18] R. Schimming, Z. f. Analysis u. ihre Anw. **2**, 71 (1982)

[19] R. Schimming, Math. Nachr. **147**, 217 (1990)